

Water Hammer and the Method of Characteristics

Introduction

IEEE Standard 1207-2004 “IEEE Guide for the Application of Turbine Governing Systems for Hydroelectric Generating Units” [1] advocates the use of computer simulation for stability studies on hydroelectric plant at an early stage in the project life-cycle. Predicting the dynamic performance of a planned installation allows design constraints to be revised before construction begins and reduces the time required for governor tuning when the system is being commissioned.

The basic simulation models for hydroelectric plant, suitable for analysis and design of control systems, are described in a well-known IEEE paper written by a group of industry experts [2]. Section 4.4 of the paper describes how travelling-wave effects in the water passages can be simulated by solving the one-dimensional ‘water hammer’ equations by means of the Method of Characteristics. The paper compares simulation results obtained by this method with simpler models. It does not, however, include a detailed description of the solution method itself.

A conceptual description of the ‘water hammer’ phenomenon and the derivation of the mathematical model as a hyperbolic partial differential equation is to be found in many sources, such as the book by Larock *et al* [3]. The solution for a typical water hammer problem turns out to be a pressure wave travelling back and forth along the conduit. An early analytical solution of the basic wave equation is due to d’Alembert and this provides a valuable conceptual explanation of the phenomena involved.

Analytical solutions prove to be impossible for just about any case of practical interest so it is necessary to use numerical methods instead. Essentially, the Method of Characteristics converts the partial differential equation into a ordinary differential equations whose validity is constrained to a set of characteristic curves, related to the velocity of the travelling wave. Typically, the method of finite differences is then applied to discretise the equations (temporally and spatially) so that computer numerical integration techniques can be applied.

Further background will not be included here; instead, we simply present a summary of the mathematics as a convenient supplement to the IEEE paper.

The water hammer equations

These are:

$$\frac{dV}{dt} + \frac{1}{\rho} \frac{\partial p}{\partial x} + g \frac{dz}{dx} + \frac{f}{2D} V|V| = 0$$
$$\rho a^2 \frac{\partial V}{\partial x} + \frac{dp}{dt} = 0$$
(1)

where:

x is the axial distance along the streamline

z is the elevation at any point of the streamline

V is the mean flow velocity

p is the pressure

a is the wave speed

ρ is the fluid density

f is the Darcy-Weisbach friction factor

D is the pipe diameter

g is the acceleration due to gravity.

These equations have only one spatial dimension (x). For liquid flowing in a pipe, the flow velocity (V) is considered to be uniform across the pipe cross-section. It is related to the volumetric flow (Q) in any section of the pipe by the cross-sectional area: $Q = AV$.

Noting that:

$$\frac{dV(x,t)}{dt} = \frac{\partial V}{\partial x} \frac{dx}{dt} + \frac{\partial V}{\partial t} = \frac{\partial V}{\partial x} V + \frac{\partial V}{\partial t} \quad (2)$$
$$\frac{dp}{dt} = \frac{\partial p}{\partial x} \frac{dx}{dt} + \frac{\partial p}{\partial t} = \frac{\partial p}{\partial x} V + \frac{\partial p}{\partial t}$$

allows (1) to be simplified by omitting the 'convective acceleration' terms, which are usually negligible in comparison with other terms, to leave:

$$\boxed{\begin{aligned} \frac{\partial V}{\partial t} + \frac{1}{\rho} \frac{\partial p}{\partial x} + g \frac{dz}{dx} + \frac{f}{2D} V|V| &= 0 \\ \rho a^2 \frac{\partial V}{\partial x} + \frac{\partial p}{\partial t} &= 0 \end{aligned}} \quad (3)$$

Further analysis will proceed based on (3).

Bernoulli Equation

The **Bernoulli equation** states that the total head is constant along a streamline:

$$\frac{p}{\gamma} + z + \frac{V^2}{2g} = K \quad (4)$$

where:

$H = \frac{p}{\gamma} + z$ is the 'piezometric head',

z is the 'elevation head',

$\frac{V^2}{2g}$ is the 'velocity head' and

$\frac{p}{\gamma}$ is the 'pressure head'.

$\gamma = \rho g$ is known as the 'specific weight' of the fluid.

The velocity head term corresponds to the convective acceleration term in (2) and in many cases of flow in pipes is negligible compared with frictional terms.

Analytical solution

Analytical solutions of (3) are restricted to simple cases.

For pipe flow, it is common practice to express (3) in terms of piezometric head (H) instead of the pressure (p). To do this, note that:

$$\begin{aligned} \therefore \frac{\partial p}{\partial x} &= \rho g \frac{\partial H}{\partial x} - \rho g \frac{\partial z}{\partial x} \\ \text{and } \frac{\partial p}{\partial t} &= \rho g \frac{\partial H}{\partial t} - \rho g \frac{\partial z}{\partial t} \end{aligned} \tag{5}$$

where the term $\frac{\partial z}{\partial t}$ is zero because the pipe elevation does not change with time and $\frac{\partial z}{\partial x}$ is alternatively expressed as $\sin(\theta)$, where θ is the angle at which the pipe slopes.

Substituting (5) into (3) allows the pipe slope term to be eliminated:

$$\begin{aligned} \frac{\partial V}{\partial t} + g \frac{\partial H}{\partial x} + \frac{f}{2D} V|V| &= 0 \\ \rho a^2 \frac{\partial V}{\partial x} + \rho g \frac{\partial H}{\partial t} &= 0 \end{aligned} \tag{6}$$

The dissipative term $V|V|$ in (6) is nonlinear (the friction depends on the sign of the velocity).

This makes an analytical solution very difficult, even for simple boundary conditions. To proceed, consider the frictionless case and take the partial derivative of (6) with respect to both t and x:

$$\begin{aligned} \rho \frac{\partial^2 V}{\partial t^2} + g \frac{\partial^2 H}{\partial t \partial x} &= 0 \\ \frac{\partial^2 V}{\partial t \partial x} + \frac{g}{a^2} \frac{\partial^2 H}{\partial t^2} &= 0 \end{aligned} \tag{7}$$

and

$$\frac{\partial^2 V}{\partial x \partial t} + g \frac{\partial^2 H}{\partial x^2} = 0 \quad (8)$$

$$a^2 \frac{\partial^2 V}{\partial x^2} + g \frac{\partial^2 H}{\partial x \partial t} = 0$$

Subtracting (8) from (7) yields:

$$\frac{\partial^2 V}{\partial t^2} - a^2 \frac{\partial^2 V}{\partial x^2} = 0 \quad (9)$$

$$\frac{\partial^2 H}{\partial t^2} - a^2 \frac{\partial^2 H}{\partial x^2} = 0$$

which are classic one-dimensional **wave equations**. These linear hyperbolic partial differential equations can be solved individually for piezometric head or flow velocity.

An analytical solution to (9) is due to d'Alembert.

$$H(x,t) = \frac{f(x-at) + f(x+at)}{2} + \frac{1}{2a} \int_{x-at}^{x+at} g(s) ds \quad (10)$$

where:

- $H(x,0)$ is the initial pressure profile across the pipe
- $g(x) = \frac{\partial H(x,0)}{\partial t}$ is the initial rate of change of pressure

The d'Alembert solution is the sum of two waves travelling in the opposite direction at the constant wave velocity 'a'. A complete solution is obtained by specifying initial conditions (at $t = 0$) and boundary conditions (at $x = 0$ and $x = L$), where the boundary conditions are either 'free' or 'fixed'.

The d'Alembert solution is extremely valuable as a concept for visualising and understanding water hammer. Its use in practice is limited to cases which have a single dominant conduit and very low levels of friction.

Practical solutions are usually obtained numerically by means of a finite difference form of the Method of Characteristics.

Numerical solution by the Method of Characteristics

The following procedure changes the two PDEs in (3) into a single ODE.

Introduce the Lagrange multiplier λ :

$$\lambda \left\{ \frac{\partial V}{\partial t} + \frac{1}{\rho} \frac{\partial p}{\partial x} + g \frac{dz}{dx} + \frac{f}{2D} V |V| \right\} + \left\{ \rho a^2 \frac{\partial V}{\partial x} + \frac{\partial p}{\partial t} \right\} = 0 \quad (11)$$

Gather terms:

$$\left(\lambda \frac{\partial V}{\partial t} + \rho a^2 \frac{\partial V}{\partial x}\right) + \left(\frac{\lambda}{\rho} \frac{\partial p}{\partial x} + \frac{\partial p}{\partial t}\right) + \lambda g \frac{dz}{dx} + \lambda \frac{f}{2D} V|V| = 0 \quad (12)$$

Now $\lambda \frac{dV(x,t)}{dt} = \lambda \frac{\partial V}{\partial t} + \lambda \frac{\partial V}{\partial x} \frac{dx}{dt}$ so, by comparison of coefficients, the first term in (12) can be made equal to $\lambda \frac{dV}{dt}$ by letting:

$$\lambda \frac{dx}{dt} = \rho a^2 \quad (13)$$

Also $\frac{dp(x,t)}{dt} = \frac{\partial p}{\partial x} \frac{dx}{dt} + \frac{\partial p}{\partial t}$ and the second term in (12) can be made equal to $\frac{dp}{dt}$ by letting:

$$\frac{\lambda}{\rho} = \frac{dx}{dt} \quad (14)$$

So to satisfy both (13) and (14) we need:

$$\lambda \cdot \frac{\lambda}{\rho} = \rho a^2 \quad \therefore \lambda^2 = \rho^2 a^2 \text{ with the solution:}$$

$$\lambda = \pm \rho a \quad (15)$$

where ρ is always positive but the wave velocity can be positive or negative.

By constraining $\lambda = \pm \rho a$, the first two terms in (12) become full differentials:

$$\lambda \frac{dV}{dt} + \frac{dp}{dt} + \lambda g \frac{dz}{dx} + \lambda \frac{f}{2D} V|V| = 0 \quad (16)$$

The PDEs (3) have been reduced to the single ODE (16) which applies at points where

$$\lambda = \frac{dx}{dt} = +\rho a \text{ and points where } \lambda = \frac{dx}{dt} = -\rho a.$$

Choosing $\lambda = \rho a$:

$$\rho a \frac{dV}{dt} + \frac{dp}{dt} + \rho a g \frac{dz}{dx} + \rho a \frac{f}{2D} V|V| = 0$$

$$\therefore \frac{dV}{dt} + \frac{1}{\rho a} \frac{dp}{dt} + g \frac{dz}{dx} + \frac{f}{2D} V|V| = 0 \quad (17)$$

which is known as the **C⁺** equation.

Similarly, choosing $\lambda = -\rho a$:

$$\frac{dV}{dt} - \frac{1}{\rho a} \frac{dp}{dt} + g \frac{dz}{dx} + \frac{f}{2D} V|V| = 0 \quad (18)$$

which is known as the C- equation.

Again, it is possible to use the piezometric head (H) instead of the pressure (p) by recalling that $p = \gamma(H - z) = \rho g(H - z)$

$$\therefore \frac{dp}{dt} = \rho g \frac{dH}{dt} - \rho g \frac{dz}{dx} \frac{dx}{dt} \quad (19)$$

Substituting in the C+ equation (17):

$$\frac{dV}{dt} + \frac{1}{\rho a} \left(\rho g \frac{dH}{dt} - \rho g \frac{dz}{dx} \frac{dx}{dt} \right) + g \frac{dz}{dx} + \frac{f}{2D} V|V| = 0$$

$$\therefore \frac{dV}{dt} + \frac{g}{a} \frac{dH}{dt} - \frac{g}{a} \frac{dz}{dx} \frac{dx}{dt} + g \frac{dz}{dx} + \frac{f}{2D} V|V| = 0$$

Now substitution of $\frac{dx}{dt} = +a$, and similarly $\frac{dx}{dt} = -a$ for the C- equation, gives:

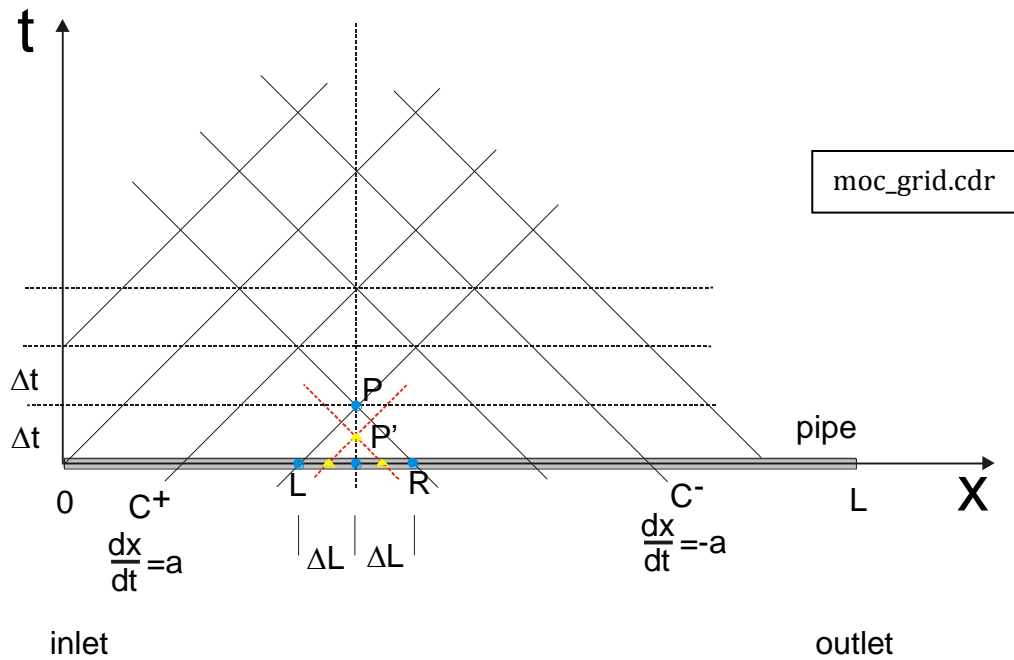
$\frac{dV}{dt} + \frac{g}{a} \frac{dH}{dt} + \frac{f}{2D} V V = 0$	C⁺	(20)
$\frac{dV}{dt} - \frac{g}{a} \frac{dH}{dt} + \frac{f}{2D} V V = 0$	C⁻	

Equations (20) are the basis for the finite-difference solution of the water hammer problem.

$$\text{For C}^+, \frac{dx}{dt} = a \quad \therefore x = at + c' \quad \therefore t = \frac{x}{a} - \frac{c'}{a} = \frac{1}{a}x - k'$$

$$\text{For C}^-, \frac{dx}{dt} = -a \quad \therefore x = -at + c'' \quad \therefore t = -\frac{x}{a} - \frac{c''}{a} = -\frac{1}{a}x - k''$$

These represent a set of straight-line relationships between t and x on which C⁺ and C⁻ are valid as shown in the diagram below.



This means that, if we wish to know the values of V and H at some selected point x_p along the length of the pipe at some time Δt , then it must be done on the basis of values of V and H at time zero at $(x - \Delta L)$ and $(x + \Delta L)$. If we wanted to work out the values of V and H at P' at (say) time $\Delta t/2$ then we would need to do this on the basis of values of V and H at time zero at $(x - \Delta L/2)$ and $(x + \Delta L/2)$.

So the smaller the time step is made, the finer the spatial grid must be and *vice versa*: $\Delta L = a\Delta t$ must hold.

This constraint means that the computation time will vary with the **square** of either the number of time-steps or spatial segments chosen.

Steady state flow

Substituting $\frac{dV}{dt} = 0$ in (20):

$$\frac{g}{a} \frac{dH}{dt} + \frac{f}{2D} V|V| = 0$$

$$\therefore \frac{g}{a} \frac{dH}{dx} \frac{dx}{dt} + \frac{f}{2D} V|V| = 0$$

$$\therefore g \frac{dH}{dx} = -\frac{f}{2D} V|V|$$

$$\therefore \frac{dH}{dx} = -\frac{f}{2gD} V|V|$$

So the gradient of the piezometric head is determined by the constant flow velocity, V_0 :

$$H(x) = -\frac{f}{2gD} V_0 |V_0| \cdot x + c_i$$

Let $H(x=0) = H_{res}$ be the head at the reservoir. Then:

$$H(0) = H_{res} = -\frac{f}{2gD} V_0 |V_0| \cdot 0 + c_i$$

$$\therefore c_i = H_{res}$$

Let $H(x=L) = H_L$ be the head at the lower end of the pipe. Then:

$$H(L) = -\frac{f}{2gD} V_0 |V_0| \cdot L + c_i = -\frac{f}{2gD} V_0 |V_0| \cdot L + H_{res}$$

$$\therefore V_0 |V_0| = \frac{2gD}{fL} (H_{res} - H_L)$$

So the steady state velocity is given by:

$$V_0 = \sqrt{\frac{2gD}{fL} (H_{res} - H_L)} \quad (21)$$

Finite difference solution for the Method of Characteristics

The advantages of the finite-difference method are:

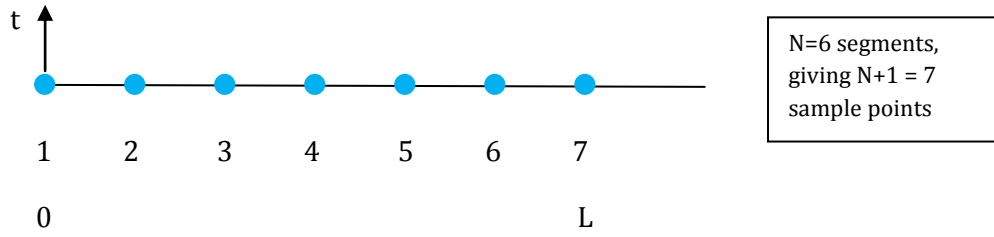
- it is relatively easy to implement
- it produces a numerical approximation to (20) on a rectangular grid in the xt plane

A disadvantage of the direct finite-difference method is that it generally does not directly incorporate the fact that information about the solution of (20) is propagated along characteristics. As a consequence, finite difference methods often produce poor results when the solution has large gradients.

The major strength of the numerical method of characteristics is that it tracks information about the solution along approximations to the characteristics. This makes the numerical method of characteristics generally better at dealing with a solution that has sharp fronts. Its disadvantages are:

- it produces an approximation on an irregularly spaced set of points in the xt plane
- it is somewhat more complicated to implement than a direct finite-difference method

Suppose we have N segments in a section of pipe, so that we have $N+1$ (spatial) sample points separated by a distance ΔL .



From (20), the C⁺ equation is:

$$\frac{dV}{dt} + \frac{g}{a} \frac{dH}{dt} + \frac{f}{2D} V|V| = 0$$

A finite difference approximation for the velocity and head, (V_p, H_p) , at point P in the diagram above is expressed as:

$$\begin{aligned} \mathbf{C}^+ : \quad & \frac{V_p - V_\ell}{\Delta t} + \frac{g}{a} \frac{(H_p - H_\ell)}{\Delta t} + \frac{f}{2D} V_\ell |V_\ell| = 0 \\ \mathbf{C}^- : \quad & \frac{V_p - V_r}{\Delta t} - \frac{g}{a} \frac{(H_p - H_r)}{\Delta t} + \frac{f}{2D} V_r |V_r| = 0 \end{aligned} \quad (22)$$

provided $\Delta L = \pm a \Delta t$.

The subscripts ℓ and r in (22) refer to the sample points to the left and right of P and one time interval Δt in the past. The relationship between ΔL and Δt constrains the solution to the characteristic lines*.

Multiplying through by Δt , equations (22) can be re-written:

$$\begin{aligned} \mathbf{C}^+ : \quad & (V_p - V_\ell) + \frac{g}{a} (H_p - H_\ell) + \frac{f \Delta t}{2D} V_\ell |V_\ell| = 0 \\ \mathbf{C}^- : \quad & (V_p - V_r) - \frac{g}{a} (H_p - H_r) + \frac{f \Delta t}{2D} V_r |V_r| = 0 \end{aligned} \quad (23)$$

Equation (23) may be solved for V_p and H_p . Let $C = \frac{g}{a}$ and $K = \frac{f \Delta t}{2D}$

$$\begin{aligned} V_p &= V_\ell - C H_p + C H_\ell - K V_\ell |V_\ell| \\ \therefore V_\ell - C H_p + C H_\ell - K V_\ell |V_\ell| - V_r - C H_p + C H_r + K V_r |V_r| &= 0 \\ \therefore H_p &= \frac{1}{2C} \left[(V_\ell - V_r) + C(H_\ell + H_r) - K(V_\ell |V_\ell| - V_r |V_r|) \right] \end{aligned}$$

In general, (V_{p_i}, H_{p_i}) at the i^{th} sample point along the length of the pipe (where $(i-1)$ and $(i+1)$ refer to the points to the left and right of P respectively) is given by:

* Computer code must build in this constraint when choosing sample time and segment length.

$$\begin{aligned}
V_{pi} &= \frac{1}{2} \left[(V_{i-1} + V_{i+1}) + \frac{g}{a} (H_{i-1} - H_{i+1}) - \frac{f \Delta t}{2D} (V_{i-1} |V_{i-1}| + V_{i+1} |V_{i+1}|) \right] \\
H_{pi} &= \frac{1}{2} \left[\frac{a}{g} (V_{i-1} - V_{i+1}) + (H_{i-1} + H_{i+1}) - \frac{a}{g} \frac{f \Delta t}{2D} (V_{i-1} |V_{i-1}| - V_{i+1} |V_{i+1}|) \right]
\end{aligned}
\tag{24}$$

Equations (24) are valid for **interior points** of the solution space, $2 \leq i \leq N$.

Boundary conditions, valid for $\forall t > 0$, must be imposed on the two points at either end.

Initial conditions must be specified for $t = 0$.

These are problem dependent. Many boundary conditions exist to describe the relationship between flow and head for various pipe topologies such as constrictions and branches.

Boundary conditions

This section considers a small selection of common boundary conditions relevant to the solution of hydraulic problems in the hydroelectric power generation industry.

Constant head reservoir at upstream end

Neglecting velocity head loss, the piezometric pressure at the inlet is fixed at the reservoir's elevation, $H_{in} = Z_{res}$.

To get the boundary condition on flow velocity, use the C- relationship in (23) for the upstream boundary:

$$V_{p1} - V_r - C(H_{p1} - H_r) + KV_r |V_r| = 0$$

Now $H_{p1} = H_{in}$, $V_r = V_2$ and $H_r = H_2$ so:

$$V_{p1} = V_2 + C(H_{in} - H_2) - KV_2 |V_2|$$

Downstream valve

Assume that the valve is controlled to close such that the flow velocity decreases linearly over the closure interval from an initial steady state value down to zero. Then the imposed boundary condition is:

$$V_{p,N+1} = V_{ss} \left(1 - \frac{t}{t_c} \right) = V_{ss} \left(1 - \frac{j \Delta t}{t_c} \right)$$

where j is the time-step index, t_c is the valve closure interval and $(N+1)$ refers to the rightmost sample point on the pipe length.

Use the C^+ relationship from (23) at the right-hand boundary:

$$V_p - V_\ell + C(H_p - H_\ell) + KV_\ell |V_\ell|$$

$$\therefore (H_p - H_\ell) = -\frac{1}{C}(V_p - V_\ell) + \frac{K}{C}V_\ell |V_\ell|$$

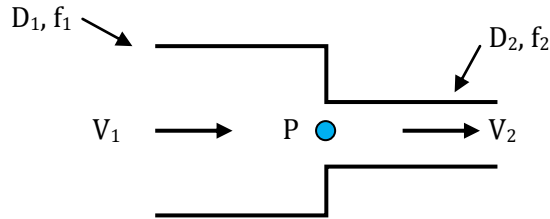
or, in terms of the boundary indices:

$$H_{p,N+1} = H_N + \frac{1}{C}(V_N - V_{p,N+1} + KV_N |V_N|)$$

Note that the value of the head is immediately available from the C^+ expression because the velocity is explicitly specified.

Change in pipe cross-sectional area

Suppose that there is a change in the cross-section of the pipe as shown here:



The pressure at P is identical for both sections if the loss at the constriction is considered negligible. So the pressure at the right-most sample point ($N+1$) of the upstream section (i) is identical to the left-most sample point (1) of the downstream section ($i+1$).[†]

$$H_{p_i,N+1} = H_{p_{i+1},1}$$

Similarly, from continuity of flow:

$$Q_{p_i,N+1} = Q_{p_{i+1},1}$$

The C^- equation for the downstream section and the C^+ equation for the upstream section are:

$$C^+ : V_p - V_\ell + C(H_p - H_\ell) + KV_\ell |V_\ell| = 0$$

$$C^- : V_p - V_r - C(H_p - H_r) + KV_r |V_r| = 0$$
(25)

Applying these gives the following set of three equations to be solved for $V_{p1,N+1}$, $V_{p2,1}$, and $H_{p1,N+1} = H_{p2,1} = H_0$.

[†] Note that this convention means that point 'N' will be just to the left of 'N+1' in the upstream section and point '2' will be just to the right of '1' in the downstream section.

$$\begin{aligned}
A_1 V_{P1,N+1} &= A_2 V_{P2,1} \\
V_{P1,N+1} - V_{1,N} + C(H_{P1,N+1} - H_{1,N}) + K_1 V_{1,N} |V_{1,N}| &= 0 \\
V_{P2,1} - V_{2,2} - C(H_{P2,1} - H_{2,2}) + K_2 V_{2,2} |V_{2,2}| &= 0
\end{aligned} \tag{26}$$

where $K_1 = \frac{f_1 \Delta t}{2D_1}$ and $K_2 = \frac{f_2 \Delta t}{2D_2}$ are the friction terms for the two pipe sections.

Re-write (26):

$$\begin{aligned}
A_1 V_{P1,N+1} &= A_2 V_{P2,1} \\
V_{P1,N+1} + CH_{P1,N+1} + \{-V_{1,N} - CH_{1,N} + K_1 V_{1,N} |V_{1,N}|\} &= 0 \\
V_{P2,1} - CH_{P2,1} + \{-V_{2,2} + CH_{2,2}\} + K_2 V_{2,2} |V_{2,2}| &= 0
\end{aligned} \tag{27}$$

and with the implied substitutions S_1 and S_2 :

$$\begin{aligned}
A_1 V_{P1,N+1} &= A_2 V_{P2,1} \\
V_{P1,N+1} + CH_0 + S_1 &= 0 \\
V_{P2,1} - CH_0 + S_2 &= 0
\end{aligned} \tag{28}$$

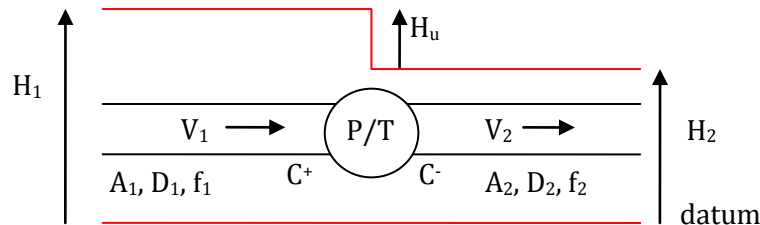
Solving (28) yields the required boundary conditions:

$$V_{P1,N+1} = -\frac{A_2(S_1 + S_2)}{(A_1 + A_2)} \quad \text{and} \quad V_{P2,1} = -\frac{A_1(S_1 + S_2)}{(A_1 + A_2)} \quad \text{and} \quad H_0 = H_{P1,N+1} = H_{P2,1} = \frac{1}{C} \left(\frac{S_2 A_2 - S_1 A_1}{A_1 + A_2} \right)$$

Note that a closed form solution of three linear simultaneous equations gives the unknown head and velocities.

Pump/turbine

The following diagram represents a variable speed pump/turbine attached to both upstream and downstream sections of pipe, of cross-sectional areas A_1 and A_2 , diameters D_1 and D_2 and friction factors f_1 and f_2 , respectively.



The pump/turbine can be modelled at different levels of complexity. Let:

$$T = f_T(Q, \omega, G) \text{ is the shaft torque on the P/T}$$

$H_u = H_1 - H_2 = f_H(Q, \omega, G)$ = change in head across the P/T as a function of flow (Q) through the P/T, its rotational speed (ω) and gate opening (G).

J = moment of inertia of P/T

$$T = J \frac{d\omega}{dt} \text{ relates shaft torque to speed}$$

The P/T characteristic is a static relationship but it is usual to assume that changes can be represented as quasi-static, i.e. a simple migration of the P/T operating point. Clearly the dynamics are more complex than this but a quasi-static model is more than adequate for this level of system simulation. CFD modelling could be used for improved accuracy but the complexity and computational burden are not warranted.

Knowing the values of V_1, V_2, H_1, H_2 and ω at time t , the goal is to calculate them at time $t+\Delta t$. From continuity it is clear that $V_1 A_1 = V_2 A_2$. The problem lies in the fact that f_T and f_H are in general nonlinear. When future values of H_{P1} and V_{P1} are calculated using equation (24), they must lie on the turbine characteristic; in effect, the operating point has changed. In general it is necessary to perform an iterative procedure to obtain consistent values of the variables at time $t+\Delta t$.

A **simpler case** to consider is the boundary condition that exists when:

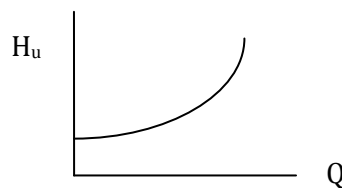
- the P/T speed is assumed constant
- one side of the P/T is connected to a constant head sump or tailrace
- the gate opening is fixed

These restrict the head across the turbine to be a function of flow only: $H_u = f_H'(Q)$.

Suppose that the head across a turbine, whose inlet is downstream of a pipe of cross sectional area A and whose outlet is connected to a constant pressure tailrace, is related to the flow through the turbine by a quadratic equation:

$$H_u = f_H'(Q) = A'_t Q^2 + B'_t Q + C'_t \quad (29)$$

where A'_t, B'_t and C'_t are constant parameters of the turbine.



If V_{P1} and H_{P1} are the velocity and head at point P at the turbine inlet at time $(t+\Delta t)$ then:

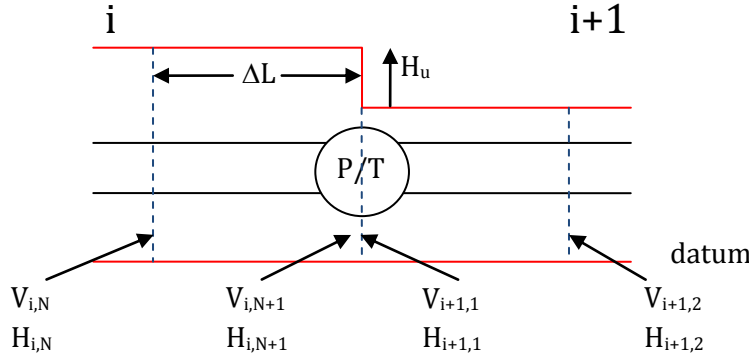
$$Q_{P1} = V_{P1} A$$

$$H_u = H_{P1} - H_{tail}$$

$$\therefore H_{P1} = A'_t A^2 V_{P1}^2 + B'_t A V_{P1} + (C'_t + H_{tail})$$

$$\therefore H_{P1} = A_t V_{P1}^2 + B_t V_{P1} + C_t$$

(30)



The point of interest (P) at the turbine inlet is downstream of the last interior node of the conduit which feeds it. So the required relationship is the (C⁺) characteristic equation in (23):

$$C^+: (V_{P_{i,N+1}} - V_{i,N}) + \frac{g}{a} (H_{P_{i,N+1}} - H_{i,N}) + KV_{i,N} |V_{i,N}| = 0 \quad (31)$$

In order to satisfy both (30) and (31) simultaneously:

$$\begin{aligned} (V_{P_{i,N+1}} - V_{i,N}) + \frac{g}{a} (A_t V_{P_{i,N+1}}^2 + B_t V_{P_{i,N+1}} + C_t) - \frac{g}{a} H_{i,N} + KV_{i,N} |V_{i,N}| &= 0 \\ \therefore \frac{gA_t}{a} V_{P_{i,N+1}}^2 + (1 + \frac{gB_t}{a}) V_{P_{i,N+1}} - \left(V_{i,N} - \frac{g}{a} C_t + \frac{g}{a} H_{i,N} - KV_{i,N} |V_{i,N}| \right) &= 0 \end{aligned} \quad (32)$$

where (32) is a quadratic equation that can be solved for $V_{P_{i,N+1}}$ and hence $H_{P_{i,N+1}}$ from (30).

By embedding the nonlinear turbine characteristic into the characteristic equations, we have enforced solutions for flow velocity and head at the boundary at time $t+\Delta t$ that are consistent with both the turbine characteristic and the hydrodynamics. The turbine nonlinearity is simple enough to lead to the quadratic equation (32) which can be solved by means of the usual formula. More complex nonlinearities will usually need an iterative solution[‡].

A **variation** that allows the gate opening (G) to be taken into account is to assume that both B'_t and C'_t in (29) are zero (which reduces the turbine characteristic to the basic orifice equation).

Then:

$$H_u = A'_t \left(\frac{Q}{GA} \right)^2 = A'_t \left(\frac{V}{G} \right)^2 \quad (33)$$

where $0 < G < 1$ is a linear modulation of the inlet area. (In practice this would be a nonlinear relationship, dependent on the type of valve or gate vane in use).

The equation corresponding to (30) is now:

$$H_{P1} = A_t \frac{V_{P1}^2}{G^2} + H_{tail} \quad (34)$$

where $A_t = A'_t$.

[‡] This leads to a difficulty if the system is to be simulated in real-time because the time required to produce an iterative solution is unbounded.

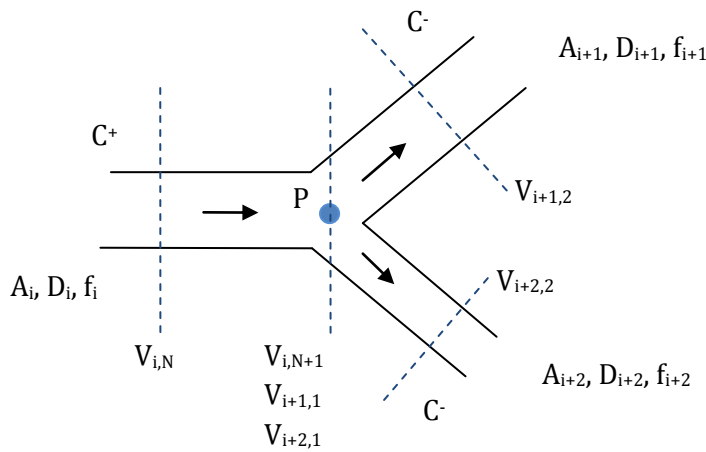
Now substituting (34) in (31) gives:

$$\begin{aligned} (V_{Pi,N+1} - V_{i,N}) + \frac{g}{a} \left(\frac{A_t}{G^2} V_{Pi,N+1}^2 + H_{tail} \right) - \frac{g}{a} H_{i,N} + KV_{i,N} |V_{i,N}| &= 0 \\ \therefore \frac{gA_t}{aG^2} V_{Pi,N+1}^2 + V_{Pi,N+1} - \left(V_{i,N} + \frac{g}{a} H_{i,N} - \frac{g}{a} H_{tail} - KV_{i,N} |V_{i,N}| \right) &= 0 \end{aligned} \quad (35)$$

At each time sample, equation (35) can be solved for $V_{Pi,N+1}$ and hence $H_{Pi,N+1}$ from (34). It is important to note that this approach leads to difficulties as $G \rightarrow 0$ because the quadratic coefficient tends to infinity.

Pipe branch

The following diagram illustrates a pipe dividing into two branches.



In general, each branch will have a different friction coefficient and diameter (hence cross-sectional area)

Continuity of flow requires that:

$$A_i V_{Pi,N+1} = A_{i+1} V_{Pi+1,1} + A_{i+2} V_{Pi+2,1} \quad (36)$$

Neglecting losses, the head at point P must be identical for all branches so:

$$H_{i,N+1} = H_{i+1,1} = H_{i+2,1} = H_0 \quad (37)$$

The characteristic equation for each branch must be satisfied:

$$\begin{aligned} V_{Pi,N+1} - V_{i,N} + C(H_{P0} - H_0) + K_i V_{i,N} |V_{i,N}| &= 0 \\ V_{Pi+1,1} - V_{i+1,2} - C(H_{P0} - H_0) + K_{i+1} V_{i+1,2} |V_{i+1,2}| &= 0 \\ V_{Pi+2,1} - V_{i+2,2} - C(H_{P0} - H_0) + K_{i+2} V_{i+2,2} |V_{i+2,2}| &= 0 \end{aligned} \quad (38)$$

The goal is to find $V_{Pi,N+1}, V_{Pi+1,1}, V_{Pi+2,1}$ and H_{P0} by solving the simultaneous equations (36) and (38). Letting:

$$\begin{aligned}
 B_i &= -\left(-V_{i,N} - CH_0 + K_i V_{i,N} |V_{i,N}|\right) \\
 B_{i+1} &= -\left(-V_{i+1,2} + CH_0 + K_{i+1} V_{i+1,2} |V_{i+1,2}|\right) \\
 B_{i+2} &= -\left(-V_{i+2,2} + CH_0 + K_{i+2} V_{i+2,2} |V_{i+2,2}|\right)
 \end{aligned} \tag{39}$$

allows (36) and (38) to be expressed in the form $[L]\underline{V} = \underline{B}$:

$$\begin{bmatrix} A_i & -A_{i+1} & -A_{i+2} & 0 \\ 1 & 0 & 0 & C \\ 0 & 1 & 0 & -C \\ 0 & 0 & 1 & -C \end{bmatrix} \begin{bmatrix} V_{Pi,N+1} \\ V_{Pi+1,2} \\ V_{Pi+2,1} \\ H_{P0} \end{bmatrix} = \begin{bmatrix} 0 \\ B_i \\ B_{i+1} \\ B_{i+2} \end{bmatrix} \tag{40}$$

which are solved as $\underline{V} = [L] \setminus \underline{B}$.

As for the case of change in cross-sectional area, a closed form solution of several linear simultaneous equations yields the unknown head and velocities.

Conclusion

The expressions summarised in the preceding sections are implemented in Matlab/Simulink and form the basis of the generic simulation package offered by GWEFR Cyf.

References

- 1 Std 1207-2004, 'IEEE Guide for the Application of Turbine Governing Systems for Hydroelectric Generating Units', New York, IEEE 2004.
- 2 IEEE Working group: 'Hydraulic turbine and turbine control models for system dynamic studies', *IEEE Trans Power Systems*, 1992, **7**, (1), pp. 167-179.
- 3 Larock, B. E., Jeppson, R. W. and Watters, G. Z.: 'Hydraulics of Pipeline Systems' 1999 (CRC Press).